STUDY MATERIALS

Suri Vidyasagar College
Department of Mathematics

SEMESTER – VI (Honours)

Course Type – DSE

Course Code – BMH6DSE43

Course Name: Mechanics – II

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Mechanics - II

* Newton's Laws of Motion :-

According to Newton, any change in the motion of an object, described with respect to a given frame of reference [Inertial frames], is the result of the mutual interaction between the object and its environment —

(i) Law of Inertia :- In an inertial frame, every free particle [i.e., a particle not acted upon by a net external force] has a constant velocity.

In a inertial system a free particle undergoes equal displacements in equal intervals of time. This fact defines a time scale or a clock for inertial frames called inertial time scale.

Motion of free particles in inertial frames will be in straight lines. For in this motion were on a curve with non-vanishing curvature, the velocity of this free particle, which is vector tangent to the path of the particle, would change with time, contradicting the first law. Thus, a path traced by a free particle in an inertial frame defines a straight line in that frame.

(ii) Law of Causality :- If the total force exerted on a particle by another objects at any specified time is represented by a vector $\vec{F}$, then

$$\vec{F} = m\vec{a} = d\vec{p}/dt \quad \text{—— (1)}$$

where $\vec{a} = d\vec{v}/dt$ is the acceleration of the particle at the given instant, m is the mass of the particle, $\vec{v}$ is the velocity of the particle at that instant and $\vec{p} = m\vec{v}$ is the linear momentum. The vector quantity $\vec{F}$ is called force and E.qn (1) above, is taken to be its definition. This law is a complete law.

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The second law is a prescription for formulating the dynamical equations of motion in inertial frames. The first law has already defined what inertial frames are. They are rectangular cartesian frames in which a free particle either stays or continues with uniform rectilinear motion ad infinitum [ad infinitum – to infinity; having no end].

(iii) Law of Reciprocity:- To the force exerted by every object on a particle, there corresponds an equal and opposite force exerted by the particle on that object.

For two interacting particles, if $\vec{F}_{21}$ is the force exerted by the first particle on the second, and $\vec{F}_{12}$ is the force exerted by the second particle on the first, we must have $\vec{F}_{12} = -\vec{F}_{21}$.

Using the second law, we have, then $\frac{d}{dt}(\vec{p}_1+\vec{p}_2) = \vec{0}$ where $\vec{p}_1$ and $\vec{p}_2$ are the linear momentum of the two particles 1 and 2 respectively. This mean that the total linear momentum $\vec{p}_1+\vec{p}_2$ is a constant of motion.

The third law says that if there exists some action on some particle, then the rest of the universe, or the remaining part of the closed system under consideration must experience the reaction.

(iv) Law of Superposition:- The total force $\vec{F}$ due to several objects acting simultaneously on a particle is equal to the vector sum of the forces $\vec{F}_K$ due to each object acting independently, that is, $\vec{F} = \sum_K \vec{F}_K$ — (2)

This is a 'divide and conquer' rule for solving mechanical problems involving complex forces. Newton did not write this law as a separate one, but it is independent of the first three laws, and was first explicitly mentioned by Daniel Bernoulli in 1738.

** Limitation of Newtonian Mechanics :-

Newtonian mech :-

→ There is no upper limit on the speed of any dynamical system.

→ Time is absolute i.e. same for all inertial frames of references.

→ Measurement can be made upto any degree of accuracy. There is no limit on the accuracy of measurement.

Limitation :-

⟨1⟩ If $v <<< c$ Then Newtonian mechanics is valid however if v is comparable with the speed of light, then Newton mechanics is failed to explain the dynamical system.

$$t_{(v)} = \frac{t_0}{\sqrt{1 - v^2/c^2}} \; ; \text{ If } v = 0 \text{ then } t_{(v)} = t_0$$

So Δt (time dilation) $= t_{(v)} - t_0 = t_0\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right)$

$\Rightarrow \Delta t = t_0(\gamma - 1)$, where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} \geq 1$

If $\gamma = 1$, $\Delta t = 0$, If $\gamma > 1$ $\Delta t \neq 0$

⟨2⟩ In very strong gravitational field General theory of relativity is used instead of Newton mechanics.

⟨3⟩ At very low dimension Quantum mechanics is used instead of Newton mechanics.

$$\Delta x . \Delta p \geq h$$

$$\Delta t \, \Delta E \geq h$$

where h = Planck's Constant.

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(7) Invariance under Galilean Transformation:-

Let S and S' be two inertial frames. Through the values of kinematical quantities will be measured differently in two frames, by definition of inertial frames, Newton's laws will be valid in both the frames, i.e.

$$m_i \frac{d^2\vec{r}_i}{dt^2} = m_i \frac{d^2\vec{r}_i{}'}{dt^2} \;(= F_i)$$

which gives $\dfrac{d^2(\vec{r}_i - \vec{r}_i{}')}{dt^2} = 0$

which can be integrated to yield

$$\vec{r}_i - \vec{r}_i{}' = \vec{u}_0 (t - t_0) = \vec{u}_0 t,$$

where u_0 is a constant velocity, the last relation follows if the two frames are made to coincide at $t = t_0$. The two frames are thus related by Galilean transformation:

$$\vec{r}_i{}' = \vec{r}_i - \vec{u}_0 t$$

How does the Lagrangian transform? If the potential energy depends only on the relative coordinates of two particles then

$$\vec{r}_2{}' - \vec{r}_1{}' = \vec{r}_2 - \vec{r}_1.$$

So that in S',

$$\begin{aligned} L' &= \tfrac{1}{2} m v'^2 - V \\ &= \tfrac{1}{2} m (\vec{v} - \vec{u}_0)^2 - V \\ &= \tfrac{1}{2} m \vec{v}^2 - V - m\vec{v}\cdot\vec{u}_0 + \tfrac{1}{2} m u_0^2 \\ &= L + \frac{d}{dt}\left[\tfrac{1}{2} m u_0^2 t - m\vec{u}_0 \cdot \vec{r}\right] \quad \text{--- (8)} \\ &= L + \frac{dF}{dt} \quad \text{--- (9)} \end{aligned}$$

where the function F is defined through (8) & (9). Note that both L and L' satisfy Euler-Lagrange equation. If L satisfies the Euler-Lagrange equation and F is any differentiable function of (q, t), then $L + \frac{dF}{dt}$ also satisfies the Euler-Lagrange equation.

This is readily proved by observing that the extra term in the Euler-Lagrange equation due to $\frac{dF}{dt}$ is

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot q_i}\left(\frac{\partial F}{\partial t}\right)\right] - \frac{\partial}{\partial q_i}\left(\frac{\partial F}{\partial t}\right) = \frac{d}{dt}\left[\frac{\partial}{\partial \dot q_i}\left(\sum_j \frac{\partial F}{\partial q_j}\dot q_j + \frac{\partial F}{\partial t}\right)\right.$$
$$\left. - \frac{\partial}{\partial q_i}\left(\sum_j \frac{\partial F}{\partial q_j}\dot q_j + \frac{\partial F}{\partial t}\right)\right]$$

Note that, $$\frac{d}{dt}\left[\frac{\partial}{\partial \dot q_i}\frac{\partial F}{\partial t}\right] = \frac{\partial}{\partial t}\left(\frac{d}{dt}\frac{\partial F}{\partial \dot q_i}\right) \quad (10)$$
$$= \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial q_i}\right)$$

Since the explicit differentiation can always be carried out at the end. further,

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot q_i}\left(\sum_j \frac{\partial F}{\partial q_j}\dot q_j\right)\right] = \frac{\partial}{\partial q_i}\left(\sum_j \frac{\partial F}{\partial q_j}\dot q_j\right)$$
$$= \sum_j \frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial q_i}\right)\dot q_j$$

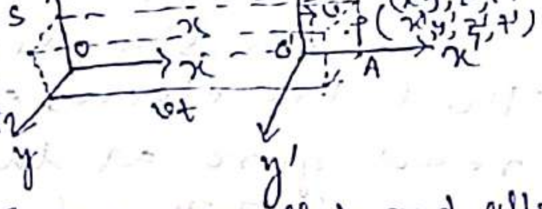
Since $\dot q_j$ is independent of the co-ordinates and the independent co-ordinates q_i and q_j are interchangeable. Using the above two relation, the RHS of (10) becomes

$$\sum_j \frac{\partial^2 F}{\partial q_j \partial q_i}\dot q_j + \frac{\partial^2 F}{\partial t \partial q_i} - \sum_j \frac{\partial^2 F}{\partial q_i \partial q_j}\dot q_j - \frac{\partial^2 F}{\partial q_i \partial t} = 0$$

which proves the assertion.

Galilean Transformation:

Galilean transformations, also called Newtonian transformations, the set of equations in classical mechanics that relate the space and time co-ordinates of two systems moving at a constant velocity relative to each other.



Let us consider now an inertial frame S and an another inertial frame S' which moves at a constant velocity v w.r.t S. We choose the three set of axis to be parallel and allow their relative motion to be along the common x, x' axis.

Let an event occur at point P whose space and time coordinates are measured in each inertial frame. An observer attached to S specifies by means of meter stick and clocks, for instance, the location and time of occurrence of this event, describing space co-ordinates x, y and z and time t to it. An another observer attached to S', using his measuring instruments specifies the same event by space-time co-ordinates x', y', z' and t'. The co-ordinates x, y, z will give the position of P relative to the origin 'O' as measured by observer S, and t will be the time of occurrence of P that observer S records with his clock. The co-ordinates x', y', z' likewise refer the position of P to the origin 'O'' and the time of P', t' to the clock of inertial observer S'.

We now ask - what the relationship is between the measurements x, y, z, t and x', y', z', t'. The two inertial observers use meter stick, which have been compared and calibrated against one another and clocks, which have been synchronised and calibrated against one another. We assume that length intervals and time intervals are absolute, that is that they are the same for all inertial observers of the same events.

For simplicity, ~~let~~ let us say that the clocks of ~~which~~ each observers read zero at the instant that the origins O and O' of the frames S and S' which are in relative motion, coincide. It is clear that

$$OA = OO' + O'A \quad \Rightarrow x = vt + x'$$

$\Rightarrow x' = x - vt$

As there is no relative motion along y & z direction so $y' = y$ and $z' = z$. As time is considered to be absolute in nature for i.e. Time remain same in all inertial frames of references so $t' = t$

$\therefore x' = x - vt, \; y' = y \; \& \; z' = z$

Hence the results.

Gibbs–Appell's Principle of Least Constraint

Willard Gibbs (1879) and later Paul Appell (1899) gave a new meaning to Lagrange's equations of motion of the first kind. Following their suggestions, let us define a quantity called the kinetic energy of acceleration of a system of N particles, given by

$$S = \frac{1}{2}\sum_{j=1}^{N} m_j |\ddot{r}_j|^2 \quad \text{——— (1)}$$

where velocity is replaced by acceleration in the usual expression for kinetic energy. Now Lagrange's equations of the first kind as given in equation (2) —

$$m_j \ddot{r}_j - F_j^{(a)} - \sum_{i=1}^{k} \lambda_i \frac{\partial h_i}{\partial \ddot{r}_j} = 0 \; ; \; j = 1, 2, \cdots, N \quad \text{——— (2)}$$

eqn (2) ⇒ (may be written as)

$$\frac{\partial}{\partial \ddot{r}_j}\left[\frac{1}{2}\sum_{j=1}^{N} m_j \ddot{r}_j^{\,2} - \sum_{j=1}^{N} F_j^{(a)} \cdot \ddot{r}_j - \sum_{i=1}^{k} \lambda_i h_i\right] = 0$$

(For whole system of particles)

$$\Rightarrow \frac{\partial G}{\partial \ddot{r}_j} = 0 \quad \text{——— (3)}$$

where,

$$G = \frac{1}{2}\sum_{j=1}^{N} m_j \ddot{r}_j^{\,2} - \sum_{j=1}^{N} F_j^{(a)} \cdot \ddot{r}_j - \sum_{i=1}^{k} \lambda_i h_i$$

i.e.

$$G = S - \sum_{j=1}^{N} F_j^{(a)} \cdot \ddot{r}_j - \sum_{i=1}^{k} \lambda_i h_i \quad \text{——— (4)}$$

is scalar point function of acceleration formed out of known quantities such as externally applied forces and constraint equations. This is called Gibbs – Appell's form of the equations of motion.

Furthermore, it is easy to check that

$$\frac{\partial^2 G}{\partial \ddot{r}_j^{\,2}} = \frac{\partial^2 S}{\partial \ddot{r}_j^{\,2}} = m_j > 0$$

.

So, the function G is such that its first derivative w.r.t acceleration $\ddot{r}_j$ is zero by requirement of the equation of motion and its second derivative w.r.t $\ddot{r}_j$ is positive as the mass of any particle is greater than zero. This means that Gibbs-Appell's form of eqns of motion is a minimum for G w.r.t all possible variations of $\ddot{r}_j$. The function G is called Gibbs-Appell's least constraint function. Gibbs-Appell's principle of least constraint states that for a given set of position vectors r_j and velocities $\dot{r}_j$, $j=1,2,\dots,N$, the Gibbs-Appell's function $G(r,\dot{r},\ddot{r})$ is a minimum if and only if the acceleration $\ddot{r}_j$ $(j=1,2,\dots N)$ are chosen to be the measured or the actual one. This effort of Gibbs and Appell can be viewed as an early attempt to geometrise dynamics.

(*) Work-Energy Relation for Constraint Forces of Sliding Friction :-

The force of friction between two surfaces depends simultaneously on how hard they are pressed against each other by a force normal to the surface of contact and on the forces of pull or push that act parallel to the surface of contact. The forces of pull or push remain exactly balanced by the forces of friction that develop at the interface, upto limits set by the co-efficient of static friction and the normal component of the pressure force. Since the point of contact does find a chance to do work on the bodies and the energy of the system remains unaffected. However, in the presence of sliding friction, the ~~mathemati~~ mechanical energy of the system gets gradually converted into heat, and therefore it is the first law of thermodynamics, rather than the simple conservation of mechanical energy that would be the most appropriate conservation law to be applied.

Suppose, a block is dragged at constant speed across a table with friction. The applied force of magnitude f acting through a distance 'd' does an amount of work $f.d$. The frictional force $\mu_K N$ ($= f$, since no acceleration is observed) does an amount of work $= -\mu_K N d = -fd$.
(where μ_K = coefficient of friction & N = normal reaction)

Thus the total work $= fd - fd = 0$.

For a point particle,

work done = change of K.E

i.e. $fd - fd = 0 = \Delta(mv^2/2)$; v = velocity

$\Rightarrow$ <u>v does not change.</u>

Moreover, for calculating the frictional work, the distance d used for the work done by the applied forces cannot be used. It is true that the block has moved through a distance d, but the frictional force at the interface has not worked through all that distance. Sherwood and Bernard (1984) suggest that d should be replaced by $d_{eff} \leq d$ for calculating the frictional work. The exact relation between d_{eff} and d should depend on the nature of the two surfaces at contact of sliding.

when two surfaces are of identical nature

$$d_{eff} = \frac{1}{2}.d ,$$

When the sliding upper block is soft and the resting lower block is very hard —

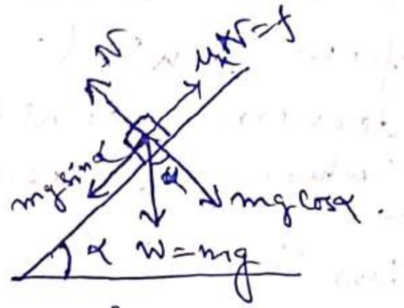
$$d_{eff} = 0, \text{ and}$$

when the sliding upper block is hard and the resting lower block is soft —

$$d_{eff} = d.$$

When a block slides through a distance 'd' down an incline (angle of inclination $= \alpha$) with friction, the block's hot teeth are continually transferring heat to the new cold regions of the incline, so the possible heat transfer from block to incline $|Q|$ is not negligible.

Newton's method would suggest the work energy relation be given by



$$(mg \sin\alpha - \mu_k N) \cdot d = \Delta \frac{1}{2} m v^2 \quad \text{—— (1)}$$

but from the first law of thermodynamics, the Sherwood equation give

$$(mg \sin\alpha) d - \mu_k N d_{eff} - |Q| = \Delta\left(\frac{1}{2} m v^2\right) + \Delta E_{\text{thermal of block}} \quad \text{—— (2)}$$

However, Eqn (2) alone is also incomplete as the effective displacement d_{eff} of the frictional force is unknown, so one has to combine the Eqn (1) and Eqn (2) giving —

$$\mu_k N (d - d_{eff}) = \Delta E_{\text{thermal of block}} + |Q|$$

Since the RHS is positive, $d_{eff} < d$.

Further, if we consider the universe as the closed system (i.e., block + incline + earth) the total change in the thermal energy of the universe would simply $\mu_k N d$.

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